

Bi-refrindexence versus bi-metricity.

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In this article we carefully distinguish the notion of bi-refrindexence (a polarization-dependent doubling in photon propagation speeds) from that of bi-metricity (where the two photon polarizations “see” two distinct metrics). We emphasise that these notions are logically distinct, though there are special symmetries in ordinary (3+1)-dimensional nonlinear electrodynamics which imply the stronger condition of bi-metricity.

To illustrate this phenomenon we investigate a generalized version of (3+1)-dimensional nonlinear electrodynamics, which permits the inclusion of arbitrary inhomogeneities and background fields. [For example dielectrics (*a lá* Gordon), conductors (*a lá* Casimir), and gravitational fields (*a lá* Landau–Lifshitz).] It is easy to demonstrate that the generalized theory is bi-refrindexent: In (3+1) dimensions the Fresnel equation, the relationship between frequency and wavenumber, is always *quartic*. It is somewhat harder to show that in some cases (*e.g.*, ordinary nonlinear electrodynamics) the quartic factorizes into two *quadratics* thus providing a bi-metric theory. Sometimes the quartic is a perfect square, implying a single unique effective metric. We investigate the generality of this factorization process.

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I. INTRODUCTION

It is commonly assumed that electromagnetic phenomena (in sharp contrast with gravitational phenomena) can be described by a linear field theory even in strong fields. But there are many reasons to expect that this is not ultimately what happens. Born and Infeld [1] proposed that some specific non-linearities could appear at strong fields preventing the existence of arbitrary large values for the electric field surrounding a charged point particle. For their part, Euler and Heisenberg [2] (see also Schwinger [3]) argued that at high values of the electric field, the quantum creation and annihilation of electron-positron pairs would give rise to an effective non-linear

electrodynamic theory. This effective non-linear electrodynamic theory can be further modified by externally driven alterations of the quantum vacuum owing to background electromagnetic [4] and gravitational fields [5], or to geometrical boundary conditions (Casimir plates, *etc.*) [6]. For some additional important examples see the references in [7]. Remarkably enough, in most cases these non-linear electrodynamic theories can be described by the use of appropriate (non-linear) media, with their associated refractive indices.

Over the last few years Professor Mário Novello and co-authors have devoted considerable effort to investigating some of the consequences of having a non-linear electrodynamic theory. On the one hand, they have been interested in the possible role of non-linear electrodynamics in circumventing the singularity problem of general relativity. In particular, they have shown that in Euler–Heisenberg electrodynamics it is possible to have a cosmology without an initial singularity [8]. On the other hand, Professor Novello and co-workers have been investigating what effects one should expect to see in the propagation of photons; now viewed as linear perturbations around a background electromagnetic configuration [9]. They were aware that in this case the causal propagation of photons is not controlled by the “physical” spacetime

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metric (or others conformally related to it), but by an “effective” metric depending on the background electromagnetic fields [10]. Professor Novello and his collaborators realized that this opened up the possibility of building “geometric structures” in a manner analogous to, but different from, the realm of usual general relativity. For arbitrary non-linear theories they have shown that black holes [11], wormholes [12] and even geometries with closed “timelike” curves [13] can be constructed. These “effective” geometries will only be felt by the photons, while other matter fields will feel the usual gravitational spacetime metric.

A very important point addressed by Professor Novello and co-workers is the quite generic appearance of bi-refringence in non-linear electrodynamics [9]. (See also the work of Plebanski [10], Dittrich and Gies [14], Schrodinger [15], and Boillat [16].) The two polarization states of the photon propagate differently. The present authors have similarly been confronted with the question of bi-refringence (or more generally multi-refringence) in general systems of second order partial differential equations (PDEs); this investigation being motivated by an abstract approach to analog relativity [17]. (The last few years have seen a proliferation of analog models of/general relativity in the literature. See [18] for an extensive reference list.) In a series of papers [17, 19] we have shown that the crucial issue in building an analog model of general relativity is the linearization of non-linear field theories around some background solution. In the present article we will apply the general analysis and language of [17] to a generalization of non-linear electrodynamics. We will show that the existence of bi-refringence is quite easily established, but that the step to bi-metricity (the existence of two different effective metrics controlling the propagation of each photon polarization) requires special conditions that are satisfied by electrodynamics in (3+1) dimensions.

We would also like to point out that nonlinear electrodynamics (in particular Born–Infeld theory) has in recent years seen a marked resurgence of interest with the advent of the notion of D-brane (see for example Polchinski [20]). Many physicists feel that D-branes will be a crucial ingredient in any final formulation of M/String theory. It happens that the motion of a D-brane in the bulk spacetime is controlled by a Born–Infeld type action [21]. This implies that while closed strings propagate following the bulk spacetime metric, open strings (whose end points are attached to D-branes) follow an effective metric derived from the Born–Infeld Lagrangian [22].

Finally, we point out that much of the formalism developed below owes a great debt to related work (by SL, Sebastiano Sonego, and MV) on photon propagation at oblique angles in the Casimir vacuum [23].

II. GENERALIZED NONLINEAR ELECTRODYNAMICS

Consider a general class of Lagrangians of the form

$$\mathcal{L}_{\text{effective}} = \mathcal{L}(F_{\mu\nu}(x), B(x)). \quad (1)$$

Here $F_{\mu\nu}$ denotes the electromagnetic field strength; and it is assumed that derivatives of this field strength do not occur in the Lagrangian. In terms of the vector potential

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2)$$

In addition $B(x)$ denotes a generic class of *external* non-dynamical background fields. These could represent, for example, a refractive index, the 4-velocity of a dielectric, the location and 4-velocity of Casimir plates or other conductors, assorted inhomogeneities and/or boundary conditions, an external gravitational field, *etc.* If these background fields are all set to their trivial position-independent values then the system reduces to ordinary nonlinear electrodynamics in which the Lagrangian depends only on the two independent Lorentz invariants that can be constructed from the field strength tensor (for example, Born–Infeld or Euler–Heisenberg electrodynamics).

The complete equations of motion for nonlinear electrodynamics consist of the Bianchi identity,

$$F_{[\mu\nu,\lambda]} = 0, \quad (3)$$

plus the dynamical equation

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \right) = 0. \quad (4)$$

We now adopt a linearization procedure: Split the electromagnetic field into an internal (possibly dynamical) background field plus a propagating photon

$$F_{\mu\nu} = F_{\mu\nu}^{\text{background}} + f_{\mu\nu}^{\text{photon}}. \quad (5)$$

Then, assuming the background satisfies the equations of motion and retaining only linear terms in the propagating photon, we have

$$(f^{\text{photon}})_{[\mu\nu,\lambda]} = 0, \quad (6)$$

and

$$\partial_\nu \left(\frac{\partial^2 \mathcal{L}}{\partial F_{\mu\nu} \partial F_{\alpha\beta}} \Big|_{\text{background}} f_{\alpha\beta}^{\text{photon}} \right) = 0. \quad (7)$$

On defining

$$\Omega^{\mu\nu\alpha\beta} = \frac{\partial^2 \mathcal{L}}{\partial F_{\mu\nu} \partial F_{\alpha\beta}} \Big|_{\text{background}}, \quad (8)$$

equation (7) can be rewritten in the somewhat more compact form

$$\partial_\alpha \left(\Omega^{\mu\alpha\nu\beta} f_{\nu\beta}^{\text{photon}} \right) = 0. \quad (9)$$

Note that the tensor $\Omega^{\mu\nu\alpha\beta}$ is symmetric with respect to exchange of the pairs of indices $\mu\nu$ and $\alpha\beta$, and antisymmetric with respect to exchange of indices within each pair. That is: $\Omega^{\mu\nu\alpha\beta}$ has most of the key symmetries of the Riemann tensor. If one wishes to work directly at the level of the linearized Lagrangian one has:

$$\mathcal{L}_{\text{linearized}} = \frac{1}{2} \Omega^{\mu\nu\alpha\beta} f_{\mu\nu}^{\text{photon}} f_{\alpha\beta}^{\text{photon}}. \quad (10)$$

The key observation is that this linearized Lagrangian generically leads to birefringence.

We now apply a restricted form of the eikonal approximation by introducing a slowly varying amplitude $f^{\mu\nu}$ and a rapidly varying phase ϕ :

$$f_{\mu\nu}^{\text{photon}} = f_{\mu\nu} e^{i\phi}. \quad (11)$$

The 4-wavevector is then defined as $k_\mu = \partial_\mu \phi$. This approximation is similar to, but not quite identical with, the usual eikonal approximation. This is because one assumes that ϕ varies on scales much smaller than those of the background, while, on the other hand, use of the Lagrangian (1) also implies that the components of k are much smaller than the values fixed by the electron mass. (This is the so-called soft-photon regime). Under these hypotheses,

$$\Omega^{\mu\alpha\nu\beta} k_\alpha f_{\nu\beta} = 0. \quad (12)$$

But in general the internal dynamical background field is itself subject to quantum fluctuations, and to take this into account the coefficients of this equation are to be identified with the expectation value of the corresponding quantum operators in the background state $|\psi\rangle$:

$$\langle \psi | \Omega^{\mu\alpha\nu\beta} | \psi \rangle k_\alpha f_{\nu\beta} = 0. \quad (13)$$

In taking this expectation value we are using the fact that the fluctuations in the internal dynamical background fields are influenced by the external non-dynamical background fields $B(x)$. (For example, in the case of the Casimir geometry, by the distance between the plates.) In the spirit of the restricted eikonal approximation there is a separation of scales between the internal dynamical background fluctuations and the propagating photon.

The Bianchi identity (6) constrains $f^{\mu\nu}$ to be of the form

$$f_{\mu\nu} = k_\mu a_\nu - k_\nu a_\mu, \quad (14)$$

where we have introduced the linearized gauge potential a for the propagating field. Inserting (14) into (13) we find

$$\langle \psi | \Omega^{\mu\alpha\nu\beta} | \psi \rangle k_\alpha k_\beta a_\nu = 0. \quad (15)$$

Note that this last equation implies that any completely antisymmetric part of $\langle \Omega^{\mu\nu\alpha\beta} \rangle$ can be discarded without affecting the equations of motion.

Of course, this entire discussion could alternatively be rephrased in terms of Hadamard's theory of the propagation of weak discontinuities [25], the formalism preferred by Professor Novello [9]. An identical equation (relating the polarization and the wavevector) is encountered.

We emphasise that the discussion in this section has (so far) been completely independent of the dimensionality of spacetime. If we now ask how many independent components arise in

$$\langle \psi | \Omega^{\mu\nu\alpha\beta} | \psi \rangle = \langle \psi | \Omega^{([\mu\nu][\alpha\beta])} | \psi \rangle \quad (16)$$

we find [in $(d+1)$ dimensions]

$$\frac{d(d+1)(d^2+d+2)}{8}. \quad (17)$$

In particular in $(3+1)$ dimensions this quantity has 21 independent components. One of these components can be taken to be the coefficient of the Levi-Civita tensor, and so does not affect the equations of motion (15). Another component can be interpreted as the overall scale of Ω , which again does not affect the equations of motion. So the number of useful independent components in Ω is 19. In contrast, two light cones only specify $2 \times 9 = 18$ components. (Two metrics would specify $2 \times 10 = 20$ components, but in this paper we are only looking at the null cones.) It is ultimately this close relationship which makes $(3+1)$ dimensions so special (and tricky).

III. FRESNEL EQUATION

Equation (15) represents a condition for a as a function of k — it constrains a to be an eigenvector, with zero eigenvalue, of the k -dependent matrix

$$A^{\mu\nu}(k) = \langle \psi | \Omega^{\mu\alpha\nu\beta} | \psi \rangle k_\alpha k_\beta. \quad (18)$$

Any non-zero solution corresponds to a physically possible field polarization, that can be identified by a unit polarization vector ϵ (provided a is not a null vector — a possibility that can always be avoided by a suitable gauge choice).

A necessary and sufficient condition for the eigenvalue problem $A^{\mu\nu} a_\nu = 0$ to have non-zero solutions is $\det(A^{\mu\nu}) = 0$; however, this gives us no information at all. Indeed, any a parallel to k is always a non-zero solution, so the condition $\det(A^{\mu\nu}) = 0$ is actually an identity. On the other hand, $a \parallel k$ is merely an unphysical gauge mode that corresponds to $f_{\mu\nu} = 0$ by (14), so we need to find other, physically meaningful, solutions of the eigenvalue problem.

To this end, we exploit gauge invariance under $a \rightarrow a + \lambda k$ and fix a gauge, thus removing the spurious modes. For the current subsection, it is particularly convenient to adopt the temporal gauge $a_0 = 0$. Then we can define a polarization vector $\epsilon_\mu \equiv a_\mu / (a_\nu a^\nu)^{1/2}$, and the eigenvalue problem $A^{\mu\nu} \epsilon_\nu = 0$ splits into the equation

$$A^{0i} \epsilon_i = 0, \quad (19)$$

plus the reduced eigenvalue problem

$$A^{ij} \epsilon_j = 0. \quad (20)$$

The latter admits a nontrivial solution only if

$$\det(A^{ij}) = 0. \quad (21)$$

The condition (21) plays the same role as the Fresnel equation in crystal optics [24] — it is a scalar equation for k and thus gives the dispersion relation for light propagating in our “medium”.

The spatial components of the matrix (18) are

$$A^{ij} = \omega^2 \langle \psi | \Omega^{i0j0} | \psi \rangle + \omega k_m \langle \psi | \Omega^{i0jm} + \Omega^{imj0} | \psi \rangle + k_m k_n \langle \psi | \Omega^{imjn} | \psi \rangle, \quad (22)$$

where $\omega = k_0$. Let us define the unit vector $\hat{k} = \vec{k}/|\vec{k}|$. Then the components of A^{ij} in a basis with one axis directed along \hat{k} are

$$\begin{aligned} A^{ij} \hat{k}_j &= \omega^2 \langle \psi | \Omega^{i0j0} | \psi \rangle \hat{k}_j + \omega k_m \langle \psi | \Omega^{imj0} | \psi \rangle \hat{k}_j \\ &= \omega \left(\omega \langle \psi | \Omega^{i0j0} | \psi \rangle \hat{k}_j + \hat{k}_m \langle \psi | \Omega^{imj0} | \psi \rangle \hat{k}_j \right) \\ &\equiv \omega V^i. \end{aligned} \quad (23)$$

In particular

$$A^{ij} \hat{k}_i \hat{k}_j = \omega^2 \langle \psi | \Omega^{i0j0} | \psi \rangle \hat{k}_i \hat{k}_j \equiv \omega^2 S. \quad (24)$$

If we now specialize for definiteness to (3+1) dimensions, then the matrix A^{ij} has the following structure

$$\begin{pmatrix} \omega^2 S & \omega V^J \\ \omega V^I & T^{IJ} \end{pmatrix}, \quad (25)$$

where I and J label the two directions orthogonal to \hat{k} in the sense of vector-space duality. (The V^I are linear in the 4-wavenumber k , while the T^{IJ} are quadratic in the 4-wavenumber k .) Evaluating the determinant by expanding in the first row or column, it is easy to see that every term will contain at least two factors of ω , which establishes

$$\det(A^{ij}) = \omega^2 \mathcal{P}_4(k), \quad (26)$$

where $\mathcal{P}_4(k)$ is a homogeneous fourth-order polynomial in the 4-wavenumber $k_\mu = (\omega, |\vec{k}| \hat{k}_i)$. While the determinant is itself a *sextic*, the physically interesting part is given by the *quartic* \mathcal{P}_4 . In fact, by the rules for partitioning determinants

$$\det(A^{ij}) = \omega^2 S \det \left[T^{IJ} - \frac{V^I V^J}{S} \right], \quad (27)$$

so that \mathcal{P}_4 is effectively a 2×2 determinant

$$\mathcal{P}_4(k) = S \det \left[T^{IJ} - \frac{V^I V^J}{S} \right], \quad (28)$$

where the elements of the 2×2 matrix are themselves quadratic in the 4-wavenumber. This allows us to rephrase the current analysis in the language of our more general “normal modes” analysis [17] by introducing a matrix $f^{\mu\nu;IJ}$ (extremely similar to but not identical with the related quantity introduced in that article) and writing

$$\mathcal{P}_4(k) = \det [f^{\mu\nu;IJ} k_\mu k_\nu]. \quad (29)$$

Here the two polarization states take on the role of (dual) “field indices” as discussed in reference [17]. The determinant is to be taken on the IJ indices.

The upshot of this analysis is that in the most general case there appear to be four dispersion relations, corresponding the four roots of the quartic equation

$$\mathcal{P}_4(k) = 0. \quad (30)$$

If we write $k = (\omega, \vec{k}) = (\omega, |\vec{k}| \hat{k})$ then two of these roots correspond to propagation in the $+\hat{k}$ direction, while the other two correspond to propagation in the $-\hat{k}$ direction.

Different polarization states are represented by linearly independent solutions of the eigenvalue problem (20), under the condition (21). Thus, the space of polarizations is exactly two-dimensional. Since equation (21) gives rise to two dispersion relations, the polarization states actually satisfy two (in general, different) eigenvalue equations,

$$\overline{A}_{(r)}^{\mu\nu} \epsilon_\nu^{(r)} = 0, \quad (31)$$

where $r = 1, 2$ labels the dispersion relations and $\overline{A}_{\mu\nu}^{(r)}$ is obtained from $A_{\mu\nu}$ by imposing the corresponding condition on k as derived from equation (30). Indeed, suppose you pick a specific 3-direction \hat{k} and have by some means determined two independent polarization states $\epsilon_\nu^{(r)}$, which are *implicitly* functions of \hat{k} and the corresponding solutions \vec{k} of $\mathcal{P}_4(k) = 0$, then one can construct a pair of two-index matrices

$$\overline{\mathcal{G}}_{(r)}^{\mu\nu} = \langle \psi | \Omega^{\mu\alpha\nu\beta} | \psi \rangle \epsilon_\alpha^{(r)} \epsilon_\beta^{(r)}. \quad (32)$$

Although these quantities are matrices with the correct index structure to be interpreted as “effective metrics” they are in the general case implicitly functions of $\bar{\omega}$, $|\vec{k}|$, and the direction \hat{k} , and so these quantities *cannot* be viewed as spacetime metrics.

It is only in some special cases (*e.g.*, ordinary nonlinear electrodynamics) that the polynomial $\mathcal{P}_4(k)$ factorizes into two quadratic forms,

$$\mathcal{P}_4(k) = (\mathcal{G}_{(1)}^{\mu\nu} k_\mu k_\nu) (\mathcal{G}_{(2)}^{\alpha\beta} k_\alpha k_\beta), \quad (33)$$

in which case we obtain two second-order dispersion relations:

$$\mathcal{G}_{(1)}^{\mu\nu} k_\mu k_\nu = 0 \quad \text{and} \quad \mathcal{G}_{(2)}^{\mu\nu} k_\mu k_\nu = 0, \quad (34)$$

with momentum-independent matrices $\mathcal{G}_{(r)}$. We *can* now interpret these two matrices \mathcal{G} as the two (inverse) effective metrics of a bi-metric theory. (More precisely they are representative elements of two conformal classes of inverse metrics, since multiplication by an arbitrary position-dependent scalar will not modify the dispersion relations.)

In very special cases not only does (33) hold, but also $\mathcal{G}_{(1)}^{\mu\nu} = \mathcal{G}_{(2)}^{\mu\nu}$. (That is, the fourth-order polynomial $\mathcal{P}_4(k)$ is a perfect square.) In this case one ends up with a single quadratic dispersion relation of the familiar form

$$\mathcal{G}^{\mu\nu} k_\mu k_\nu = 0, \quad (35)$$

where $\mathcal{G}_{\mu\nu}$ is some symmetric tensor, which we shall call the (inverse) effective metric (again defined only up to an arbitrary conformal factor).

We now wish to investigate the conditions under which these factorization (bi-metric) and uniqueness properties hold. Since the entire formalism ultimately derives from the tensor $\Omega^{\mu\alpha\nu\beta}$, we shall look for suitable algebraic constraints on this tensor.

IV. SINGLE EFFECTIVE METRIC

In the case of a single effective metric we have

$$\mathcal{G}^{\mu\nu} k_\mu k_\nu = 0. \quad (36)$$

It should be clear from our previous discussion that a necessary condition for this to happen is the absence of birefringence. We see that the wave vector is now null with respect to this (unique) “effective metric” $\mathcal{G}^{\mu\nu}$, which therefore defines an effective geometry for the propagation of light.

We warn the reader that even when a unique (inverse) effective metric $\mathcal{G}^{\mu\nu}$ is defined, we always raise and lower indices using the flat Minkowski metric $\eta_{\mu\nu}$, or in the presence of a gravitational field the physical spacetime metric $g_{\mu\nu}$. The effective metric itself, denoted $\mathbf{g}_{\mu\nu}$, is the matrix inverse of $\mathcal{G}^{\mu\nu}$. Because of the way indices are raised and lowered using the physical metric you cannot use index placement to distinguish $\mathcal{G} = \mathbf{g}^{-1}$ from \mathbf{g} .

This single effective metric situation implies that (up to possibly a piece proportional to the Levi-Civita tensor) the tensor $\langle\psi|\Omega^{\mu\alpha\nu\beta}|\psi\rangle$ must be algebraically constructible solely in terms of $\mathcal{G}^{\mu\nu}$. In view of the symmetries of $\Omega^{\mu\alpha\nu\beta}$ we know, without need for detailed calculation, that it must be of the form

$$\langle\psi|\Omega^{\mu\alpha\nu\beta}|\psi\rangle = \Psi(\mathcal{G}^{\mu\nu} \mathcal{G}^{\alpha\beta} - \mathcal{G}^{\mu\beta} \mathcal{G}^{\nu\alpha}) + \Phi \epsilon^{\mu\alpha\nu\beta} \quad (37)$$

for some quantities Ψ and Φ . By appealing to the conformal invariance of the null cones we can always absorb a factor of $2\sqrt{|\Psi|}$ into the inverse metric \mathcal{G} and so rewrite this as

$$\langle\psi|\Omega^{\mu\alpha\nu\beta}|\psi\rangle = \pm \frac{(\mathcal{G}^{\mu\nu} \mathcal{G}^{\alpha\beta} - \mathcal{G}^{\mu\beta} \mathcal{G}^{\nu\alpha})}{4} + \Phi \epsilon^{\mu\alpha\nu\beta}. \quad (38)$$

Conversely, if $\Omega^{\mu\alpha\nu\beta}$ is of the form (38), then the matrix (18) is

$$A^{\mu\nu} \propto \left\{ \mathcal{G}^{\mu\nu} (\mathcal{G}^{\alpha\beta} k_\alpha k_\beta) - (\mathcal{G}^{\mu\alpha} k_\alpha) (\mathcal{G}^{\nu\beta} k_\beta) \right\}, \quad (39)$$

and the photon propagation equation, $A^{\mu\nu} a_\nu = 0$, becomes

$$(\mathcal{G}^{\alpha\beta} k_\alpha k_\beta) \mathcal{G}^{\mu\nu} a_\nu - (\mathcal{G}^{\alpha\beta} a_\alpha k_\beta) \mathcal{G}^{\mu\nu} k_\nu = 0. \quad (40)$$

This equation is obviously satisfied by the uninteresting gauge modes $a \parallel k$, with no constraints on k . Solutions corresponding to a non-vanishing $f^{\mu\nu}$ exist only if the coefficient of $\mathcal{G}^{\mu\nu} a_\nu$ is zero, *i.e.*, if (35) holds. Thus, the two polarization states propagate with the same dispersion relation (35), and there is no birefringence.

Substituting (35) back into the propagation equation (40) we find another relationship typical of this case,

$$\mathcal{G}^{\mu\nu} k_\mu a_\nu = 0. \quad (41)$$

Formally, the above equation looks like a gauge condition. This might seem puzzling, because nowhere in the present subsection have we fixed a gauge. In fact, (41) is a consequence of the dynamical equation $A^{\mu\nu} a_\nu = 0$, when the “on-shell” condition (35) is satisfied, and it does *not* imply any gauge fixing.

It is also interesting to notice that there is now a self-consistency or “bootstrap” condition,

$$\pm \frac{3}{4} \mathcal{G}^{\mu\nu} = \langle\psi|\Omega^{\mu\alpha\nu\beta}|\psi\rangle \mathbf{g}_{\alpha\beta}. \quad (42)$$

We stress that these relations depend only on the assumed existence of a single unique effective metric $\mathbf{g}_{\mu\nu}$ — they do not make any reference to other specifics of the external background fields $B(x)$ or the quantum state.

Finally we point out that in this mono-refractive case the linearized Lagrangian reduces to

$$\begin{aligned} \mathcal{L}_{\text{linearized}} = & \pm \frac{1}{4} \mathcal{G}^{\beta\mu} f_{\mu\nu}^{\text{photon}} \mathcal{G}^{\nu\alpha} f_{\alpha\beta}^{\text{photon}} \\ & + \frac{1}{2} \Phi \epsilon^{\mu\nu\alpha\beta} f_{\mu\nu}^{\text{photon}} f_{\alpha\beta}^{\text{photon}}. \end{aligned} \quad (43)$$

With hindsight, this is exactly what we should have expected. If we now use this Lagrangian formulation to demand positivity of energy [a feature missing from the purely kinematical analysis based on equation (15)] then we should set $\pm \rightarrow +1$. Note that we have also used the conformal invariance of the null cones to normalize \mathcal{G} in the conventional manner. Finally the Φ term is simply the well-known Pontryagin index.

V. ORDINARY NONLINEAR ELECTRODYNAMICS

In order to see how to develop a general ansatz that leads to bi-metricity (perhaps not the most general

ansatz) it is useful to consider the explicit form of the tensor $\Omega_{\mu\nu\alpha\beta}$ for *ordinary* nonlinear electrodynamics:

$$\mathcal{L}_{\text{NLE}} = \mathcal{L}(\mathcal{F}, \mathcal{G}). \quad (44)$$

Here we have adopted the now common variables [14]

$$\mathcal{F} \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\vec{B}^2 - \vec{E}^2), \quad (45)$$

$$\mathcal{G} \equiv \frac{1}{4} F_{\mu\nu} {}^*F^{\mu\nu} = -\vec{E} \cdot \vec{B}. \quad (46)$$

For such Lagrangians

$$\begin{aligned} \Omega^{\mu\nu\alpha\beta} &= \frac{1}{4} (\partial_{\mathcal{F}} \mathcal{L}) (\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\beta} \eta^{\nu\alpha}) \\ &+ \frac{1}{4} (\partial_{\mathcal{G}} \mathcal{L}) \epsilon^{\mu\nu\alpha\beta} \\ &+ F^{\mu\nu} F^{\alpha\beta} (\partial_{\mathcal{F}}^2 \mathcal{L}) + {}^*F^{\mu\nu} {}^*F^{\alpha\beta} (\partial_{\mathcal{G}}^2 \mathcal{L}) \\ &+ (F^{\mu\nu} {}^*F^{\alpha\beta} + {}^*F^{\mu\nu} F^{\alpha\beta}) \partial_{\mathcal{F}\mathcal{G}} \mathcal{L}. \end{aligned} \quad (47)$$

As soon as one inserts this tensor into the photon equation of motion (12), the completely antisymmetric part proportional to the Levi-Civita tensor drops out, because of the Bianchi identity (6). The remaining pieces reproduce the photon equation of motion in the perhaps more usual form considered by Dittrich and Gies [14], or Novello and co-workers [9].

Unless one has a specific need to perform calculations to orders higher than $O(\alpha^2)$, it is often sufficient to consider the Euler-Heisenberg Lagrangian [2] which, in the \mathcal{F} - \mathcal{G} formalism adopted above, takes the form

$$\mathcal{L}_{\text{EH}} = -\frac{1}{4\pi} \mathcal{F} + c_1 \mathcal{F}^2 + c_2 \mathcal{G}^2, \quad (48)$$

with

$$c_1 = \frac{\alpha^2}{90\pi^2 m_e^4}, \quad c_2 = \frac{7\alpha^2}{360\pi^2 m_e^4}. \quad (49)$$

The terms proportional to \mathcal{F}^2 and \mathcal{G}^2 of this Lagrangian are quartic in the field, and describe the low-energy limit of the box diagram in QED, when four photons couple to a single virtual electron loop. Thus, the Lagrangian (48) is only accurate to order α^2 , and it is meaningless to retain higher order terms within this model. For the Euler-Heisenberg Lagrangian, the tensor $\Omega^{\mu\nu\alpha\beta}$ is

$$\begin{aligned} \Omega^{\mu\nu\alpha\beta} &= \left(-\frac{1}{16\pi} + \frac{c_1 \mathcal{F}}{2} \right) (\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\beta} \eta^{\nu\alpha}) \\ &+ \frac{c_2 \mathcal{G}}{2} \epsilon^{\mu\nu\alpha\beta} \\ &+ \frac{c_1}{2} F^{\mu\nu} F^{\alpha\beta} + \frac{c_2}{2} {}^*F^{\mu\nu} {}^*F^{\alpha\beta}. \end{aligned} \quad (50)$$

Again, when one inserts this tensor into the photon equation of motion (12), the completely antisymmetric part proportional to the Levi-Civita tensor drops out.

Suppose we now adopt the “rotated” quantity

$$K = \cos \theta F + \sin \theta {}^*F, \quad (51)$$

so that

$${}^*K = -\sin \theta F + \cos \theta {}^*F. \quad (52)$$

Then for both (ordinary) NLE and Euler-Heisenberg electrodynamics we can put Ω into the form

$$\begin{aligned} \Omega^{\mu\nu\alpha\beta} &= \Psi (\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\beta} \eta^{\nu\alpha}) \\ &+ \Phi \epsilon^{\mu\nu\alpha\beta} \\ &+ a K^{\mu\nu} K^{\alpha\beta} + b {}^*K^{\mu\nu} {}^*K^{\alpha\beta}. \end{aligned} \quad (53)$$

The matrix $A^{\mu\alpha}$ is then [suppressing explicit indices]

$$\begin{aligned} A &= \Psi \{ \eta(k, k) \eta - (\eta k) \otimes (\eta k) \} \\ &+ a (K k) \otimes (K k) + b ({}^*K k) \otimes ({}^*K k). \end{aligned} \quad (54)$$

Note that because both K and *K are antisymmetric $K(k, k)$ and ${}^*K(k, k) = 0$; therefore $A k = 0$ as expected. [We have defined $\eta(k, k) \equiv \eta^{\mu\nu} k_\mu k_\nu$, $K(k, k) \equiv K^{\mu\nu} k_\mu k_\nu$, *etc.*] Now consider $\det'(A)$, the “reduced” determinant in the three directions orthogonal to k . We adopt this particular reduced determinant as a technically convenient alternative to using the temporal gauge. To be precise we define

$$\det'(A) = \left. \frac{d[\det(A + \epsilon I)]}{d\epsilon} \right|_{\epsilon=0}. \quad (55)$$

which implies

$$\det' A = \frac{1}{3!} \{ [\text{Tr}(A)]^3 + 2[\text{Tr}(A^3)] - 3[\text{Tr}(A)][\text{Tr}(A^2)] \}. \quad (56)$$

Now use this formula, or the fact that in 3 dimensions

$$\begin{aligned} \det_3(\lambda I + u \otimes u + v \otimes v) \\ = \lambda \{ \lambda^2 + \lambda(u^2 + v^2) + [u^2 v^2 - (u \cdot v)^2] \} \end{aligned} \quad (57)$$

(useful when k is timelike with respect to η), to deduce

$$\begin{aligned} \det'(A) &= -\Psi I(k, k) \left\{ \Psi^2 \eta(k, k)^2 \right. \\ &- \Psi \eta(k, k) [a \gamma(Kk, Kk) + b \gamma({}^*Kk, {}^*Kk)] \\ &+ ab [\gamma(Kk, Kk) \gamma({}^*Kk, {}^*Kk) \\ &\left. - \gamma(Kk, {}^*Kk)^2] \right\}. \end{aligned} \quad (58)$$

[The presence of the $-$ sign arises from the indefinite nature of the metric η . We have also verified the above formulae via explicit evaluation of the determinants using Maple. Note that we now distinguish between the (inverse) contravariant Minkowski metric η and the covariant Minkowski metric $\gamma = \eta^{-1}$. The wavevector k is always taken to be covariant while both K and *K are assumed doubly contravariant. We have defined $\gamma(Kk, Kk) \equiv \gamma_{\mu\nu} (Kk)^\mu (Kk)^\nu$, *etc.*]

Discarding the uninteresting factor of $I(k, k) = k^T k$ (it corresponds to the factor ω^2 encountered when we

worked in temporal gauge) we identify

$$\begin{aligned} \mathcal{P}_4(k) = & \Psi^2 \eta(k, k)^2 \\ & - \Psi \eta(k, k) [a \gamma(Kk, Kk) + b \gamma(*Kk, *Kk)] \\ & + ab[\gamma(Kk, Kk) \gamma(*Kk, *Kk) \\ & - \gamma(Kk, *Kk)^2]. \end{aligned} \quad (59)$$

This is clearly a quartic in k , and the “miracle” of (ordinary) nonlinear electrodynamics is that it factorizes into two quadratics. To establish this factorization we use some very special properties of (3+1) dimensions:

$$T_{\text{Maxwell}} = F \gamma F - \frac{1}{4} \text{tr}(F \gamma F \gamma) \eta \quad (60)$$

$$= *F \gamma *F - \frac{1}{4} \text{tr}(*F \gamma *F \gamma) \eta \quad (61)$$

$$= *F \gamma *F + \frac{1}{4} \text{tr}(F \gamma F \gamma) \eta \quad (62)$$

and

$$F \gamma *F = \mathcal{G} \eta = *F \gamma F. \quad (63)$$

In terms of the “rotated variables” the Maxwell stress energy tensor is given by

$$T_{\text{Maxwell}} = K \gamma K - \frac{1}{4} \text{tr}(K \gamma K \gamma) \eta \quad (64)$$

$$= *K \gamma *K - \frac{1}{4} \text{tr}(*K \gamma *K \gamma) \eta \quad (65)$$

$$= *K \gamma *K + \frac{1}{4} \text{tr}(K \gamma K \gamma) \eta. \quad (66)$$

Additionally

$$K \gamma *K = \mathcal{G}_\theta \eta = *K \gamma K, \quad (67)$$

where

$$\mathcal{F}_\theta = \cos(2\theta) \mathcal{F} + \sin(2\theta) \mathcal{G} \quad (68)$$

$$\mathcal{G}_\theta = -\sin(2\theta) \mathcal{F} + \cos(2\theta) \mathcal{G}. \quad (69)$$

From these relations we can deduce

$$\gamma(Kk, Kk) = -T_{\text{Maxwell}}(k, k) - \mathcal{F}_\theta \eta(k, k), \quad (70)$$

$$\gamma(*Kk, *Kk) = -T_{\text{Maxwell}}(k, k) + \mathcal{F}_\theta \eta(k, k), \quad (71)$$

$$\gamma(Kk, *Kk) = -\mathcal{G}_\theta \eta(k, k). \quad (72)$$

When substituted into \mathcal{P}_4 this implies (dropping the explicit “Maxwell” subscript)

$$\mathcal{P}_4 = a_0 \eta(k, k)^2 + a_1 \eta(k, k) T(k, k) + a_2 T(k, k)^2, \quad (73)$$

for suitable a_0, a_1, a_2 . Indeed

$$a_0 = \Psi^2 + \Psi(a - b)\mathcal{F}_\theta - ab(\mathcal{F}_\theta^2 + \mathcal{G}_\theta^2), \quad (74)$$

$$a_1 = \Psi(a + b), \quad (75)$$

$$a_2 = ab. \quad (76)$$

The quartic will factorize provided

$$a_0 + a_1 x + a_2 x^2 = 0 \quad (77)$$

has a solution in the real numbers. Fortunately the discriminant is easily seen to be a sum of squares and so is always positive.

$$a_1^2 - 4a_0 a_2 = [\Psi(a - b) + 2ab\mathcal{F}_\theta]^2 + [2ab\mathcal{G}_\theta]^2 \geq 0. \quad (78)$$

This is now enough to guarantee that \mathcal{P}_4 factorizes, indeed

$$\mathcal{P}_4 = a_0 [\eta(k, k) + b_1 T(k, k)] [\eta(k, k) + b_2 T(k, k)] \quad (79)$$

for suitable a_0, b_1, b_2 . These coefficients will be functions of $\Psi, a, b, \mathcal{F}_\theta$ and \mathcal{G}_θ whose precise form is not needed for the point we are currently making: *As long as the Lagrangian is only a function of the two invariants \mathcal{F} and \mathcal{G} , then the theory is not just birefringent, it is truly bi-metric.*

VI. A GENERAL BI-METRIC ANSATZ

Based on the above we can now guess a general ansatz for $\Omega^{\mu\nu\alpha\beta}$ that always leads to bi-metric propagation (we do not guarantee that this is the most general ansatz). Let

$$\begin{aligned} \Omega^{\mu\nu\alpha\beta} = & \Psi (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) \\ & + \Phi \epsilon^{\mu\nu\alpha\beta} \\ & + a K^{\mu\nu} K^{\alpha\beta} + b *K^{\mu\nu} *K^{\alpha\beta}, \end{aligned} \quad (80)$$

where $g^{\mu\nu}$ is now any symmetric matrix of Lorentzian signature and $K^{\mu\nu}$ is any anti-symmetric matrix — in particular K does not necessarily have anything to do with the electromagnetic field. All the analysis in the previous section can now be converted into purely algebraic statements about g and K . For example T no longer has the interpretation of being the Maxwell stress-energy tensor, it is simply an algebraic matrix defined by

$$T = K g^{-1} K - \frac{1}{4} \text{tr}(K g^{-1} K g^{-1}) g. \quad (81)$$

Reinterpreting everything in this purely algebraic manner, and repeating the analysis of the previous section, we see that (80) leads to bi-metric propagation with the two light cones being given by linear combinations of g and T .

Though rather general, this is not likely to be the most general bi-metric ansatz. To see this note that (80) above appears to contain $10(g) + 6(K) + 4(\Psi, \Phi, a, b) = 20$ free parameters. But Ψ can be absorbed by redefining g , while Φ does not affect the equations of motion, a can be absorbed by redefining K , and an overall scale factor does not affect the equations of motion. This leaves 16 physically interesting free parameters in (80) to be compared with $2 \times 9 = 18$ free parameters encoded in a pair

of light cones. We have not as yet been able to deduce the most general form of $\Omega^{\mu\nu\alpha\beta}$ compatible with bi-metricity.

In terms of the linearized Lagrangian our bi-metric ansatz (80) corresponds to

$$\begin{aligned}\mathcal{L}_{\text{linearized}} = & \frac{1}{2} \Psi \text{tr} (f_{\text{photon}} g^{-1} f_{\text{photon}} g^{-1}) \\ & + \frac{1}{2} \Phi \text{tr} (f_{\text{photon}} g^{-1} {}^* f_{\text{photon}} g^{-1}) \\ & + \frac{1}{2} a \text{tr} (K g^{-1} f_{\text{photon}} g^{-1})^2 \\ & + \frac{1}{2} b \text{tr} ({}^* K g^{-1} f_{\text{photon}} g^{-1})^2. \quad (82)\end{aligned}$$

VII. BIREFRINGENCE WITHOUT BI-METRICITY

To wrap up, can we now give a simple explicit example of a model that is birefringent *without* being bi-metric? Suppose we have a pair of two-forms J_1 and J_2 with $J_1 \not\propto {}^* J_2$. Then consider (as a particular example)

$$\begin{aligned}\Omega^{\mu\nu\alpha\beta} = & \Psi (\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\beta} \eta^{\nu\alpha}) \\ & + \Phi \epsilon^{\mu\nu\alpha\beta} \\ & + a J_1^{\mu\nu} J_1^{\alpha\beta} + b J_2^{\mu\nu} J_2^{\alpha\beta}. \quad (83)\end{aligned}$$

In this case, because you are now not satisfying the special algebraic constraints of the previous section, there is no reason for the determinant to factorize. Indeed

$$\begin{aligned}\mathcal{P}_4(k) = & \Psi^2 \eta(k, k)^2 \\ & - \Psi \eta(k, k) [a \gamma(J_1 k, J_1 k) + b \gamma(J_2 k, J_2 k)] \\ & + ab [\gamma(J_1 k, J_1 k) \gamma(J_2 k, J_2 k) \\ & - \gamma(J_1 k, J_2 k)^2]. \quad (84)\end{aligned}$$

In terms of the linearized Lagrangian (suppressing the inverse Minkowski metric $\gamma = \eta^{-1}$ as being understood)

$$\begin{aligned}\mathcal{L}_{\text{linearized}} = & \frac{1}{2} \Psi \text{tr} (f_{\text{photon}}^2) + \frac{1}{2} \Phi \text{tr} (f_{\text{photon}} {}^* f_{\text{photon}}) \\ & + \frac{1}{2} \{a \text{tr} (J_1 f_{\text{photon}})^2 + b \text{tr} (J_2 f_{\text{photon}})^2\}. \quad (85)\end{aligned}$$

The key point here is that the linearized Lagrangian is *not* a simply a function of the invariants \mathcal{F} and \mathcal{G} . In going to generalized nonlinear electrodynamics we have permitted additional structure in the form of external fields and boundary conditions. Unless these external fields satisfy rather particular conditions (*e.g.*, in the present example,

$J_1 \propto {}^* J_2$) there is no reason to believe the Fresnel determinant factorizes, and no reason to expect a bi-metric theory.

Indeed, if one picks a tensor Ω “at random” and explicitly evaluates $P_4(k)$ (using a symbolic program such as Maple) one rapidly concludes that arranging factorization (and so bi-metricity) is not an easy task. Bi-metricity is not generic in the set of all birefringent theories.

VIII. CONCLUSIONS

In this Festschrift article we have discussed, on quite general grounds, the phenomenon of bi-refrindexence and bi-metricity in [generalized] (3+1)-dimensional nonlinear electrodynamics. Our treatment encompass any non-linear electrodynamic theory in interaction with an arbitrary number of external non-dynamical fields characterizing a general medium (for example, a flowing dielectric, external gravitational fields, moving Casimir plates, *etc.*). We have seen that the phenomenon of bi-refrindexence is both generic and easily established in these theories. In (3+1) dimensions, the Fresnel equation is quartic and in special cases (*e.g.*, ordinary nonlinear electrodynamics) factorizes into two quadratics (*i.e.*, two metrics), typically different from each other. In more specialized situations these two metrics can be identical, leaving no opportunity for bi-refrindexence.

If we consider generalized nonlinear electrodynamics, then because of the presence of additional background fields, the close link between bi-refrindexence and bi-metricity can be broken — in such situations the Fresnel equation is intrinsically *quartic* and to naturally describe the geometry one would need to go beyond the notion of Lorentzian geometry and instead introduce the notion of a pseudo-Finsler geometry as described in [17].

In closing we emphasise that the use of nonlinear extensions to electrodynamics is currently becoming ubiquitous. The implied notions of birefrindexence, bi-metricity, and pseudo-Finsler geometries will doubtless continue to attract considerable attention. Professor Novello’s work on effective geometries and birefrindexence will continue to have important repercussions down the road.

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